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DISCRETE-TIME FEEDBACK STABILIZATION

Summary. This paper presents an algorithm for designing dynamic compensator for infinite-dimensional systems with use of finite dimensional approximation. The proposed method was then implemented in order to find the control function for thin rod heating process. The optimal sampling time was found depending on discrete output measurements.

DYSKRETNE STABILIZUJĄCE SPRZĘŻENIE ZWROTNE

Streszczenie. Artykuł prezentuje algorytm projektowania kompensatora dynamicznego dla układów nieskończenie wymiarowych wykorzystując skończenie wymiarową aproksymację. Następnie, zaproponowany algorytm został zaimplementowany do sterowania procesem nagrzewania pręta oraz doboru optymalnego kroku dyskretyzacji kompensatora przy założeniu dyskretnego pomiaru na wyjściu.

1. Introduction

One of the main areas of automatic control is connected with stabilization problems. Usually, in real time application, an algorithm consisting of two stages is used: 1. Bring the system to the valid region of linearization. 2. Stabilize the system using linear approximation. This approach is justified by topological similarity of nonlinear system and its linearization (valid only for hyperbolic systems without purely imaginary eigenvalues).

Feedback design (design of the stabilizing controller) depends on the system's form (usually we have either differential equations or transfer function for time independent systems).

The design of finite dimensional feedback is useful due to multiple reasons: 1. It is possible to use simple, finite-dimensional methods, e.g., Lyapunov functions and in consequence, Lyapunov equations strictly linked with algebraic Riccati equations. 2. Some of the systems have predefined structure, e.g., the hoisting machine (long line is a distributed system, and the drive may be modeled with finite-dimensional system).

The design of finite-dimensional controllers for infinite systems with finite set of unstable modes (or at least weakly damped ones) is widely analyzed in literature. This class of the systems was described by Triggiani (1975), or even earlier by Fattorini (1967). Using small disturbance methods and building appropriate invariant sets, Schumacher (1981, 1983) proposed finite dimensional stabilizing controllers for distributed and delayed systems. Similar results were obtained by Curtain (1984) for parabolic systems with infinite input-output operators. Also the works of Curtain and Salomon (1986), and Sakawa (1983, 1984, 1985) are worth noticing. Balas (1983) proposed a finite dimensional dynamic compensator for finite dimensional approximations of infinite systems. Similar methods were proposed by Kobayashi (1983). Gibson (1981) used finite dimensional approximation of algebraic Riccati equation. The detailed description of those works was done, e.g., by Mitkowski (1991) with 229 books and articles analyzed.

The design of stabilizing controllers is still an interesting problem (see, e.g. Przyluski (2014)), especially as there are more efficient numerical tools. Thanks to computers, nowadays, we can analyze complex mathematical models, e.g. of non-integer order Podlubny (1999), Das (2008), Caponetto (2010), Kaczorek (2011), Skruch (2013), Obrączka (2014) which sometimes better describe the real system.

In this work, we focused on an algorithm of stabilization of linear infinite dimensional system with finite set of instable modes (weakly damped) using finite discrete stabilization. As an example, we used diffusion equation which models the heating process of a thin rod.

2. Problem description

A simplified model of feedback system S (with continuous time) is depicted in the figure 1.

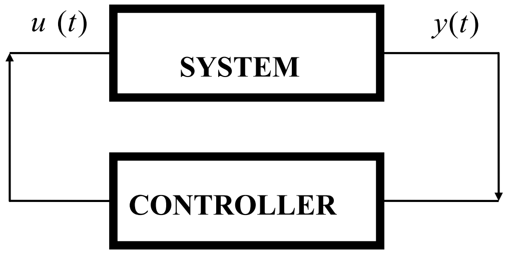

Fig. 1. Closed-loop system

Finite dimensional stabilization problem: for a given infinite system *S* find a stabilizing controller (finite dimensional) such that the closed-loop system is exponentially stable with predefined damping coefficient.

In digital control, it is necessary to use a discrete system (computer or other device with discrete time). In order to use a discrete stabilizing controller in continuous time system, we need to use the system (see fig 2.) in form of a series of pulser, continuous system S, and ZOH with input *u(k)* and output *y(k)*, *k=1,2,3,…*

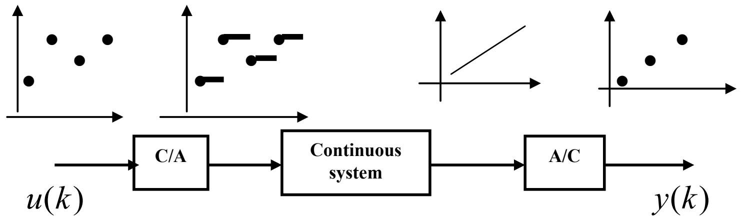

Fig. 2. Continuous-discrete system

If the pulser and ZOH work synchronously with time step $h>0$, then the parameters of discrete system S^d denoted for simplicity with A, B, C are given by the formulas calculated on the basis of continuous system:

$$A := e^{Ah}, \quad B := \int_0^h e^{At} B dt, \quad C := C \tag{1}$$

For a valid controller (both continuous and discrete), we need the controllability and observability of continuous system S. The conditions for time step h>0 which guarantee that the discrete system is also controllable and observable are known and may be found, e.g., in Mitkowski (1991, p. 141).

3. The decomposition of the system

There is a group of infinite dimensional systems which can be stabilized with use of finite dimensional methods. Let us now consider a system

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \\ &x(t) \in X, \quad u(t) \in U, \quad y(t) \in Y \end{aligned} . \tag{2}$$

For further use we will denote it as S(A,B,C). Let us now assume that (2) fulfills the following conditions:

- X, Y, U- Hilbert spaces, $\dim U < +\infty, \quad \dim Y < +\infty.$
- A is an infinitesimal generator C_0 of semi-group $T_A(t)$, for $t \geq 0$ in X
- $B \in L(U, X)$, $C \in L(X, Y)$ are bounded
- A is a discrete operator with finite number of eigenvalues with $\operatorname{Re} s > \beta$, $\beta < +\infty$

Taking into account the conditions above, we can decompose (2) into (Triggiani 1975):

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} u(t), \tag{3}$$

$$\begin{aligned} &y(t) = C_1 x_1(t) + C_2 x_2(t) + C_3 x_3(t), \\ &x_i(t) \in X_i, \ i = 1,2,3, \quad X = X_1 + X_2 + X_3, \\ &\dim X_1 < +\infty, \quad \dim X_2 = p < +\infty. \end{aligned} \tag{4}$$

The spectrum of (2) is depicted in the figure 3. The operator A_1 is responsible for unstable (or weakly damped) part of the system. The operators A_1 and A_2 are exponentially stable.

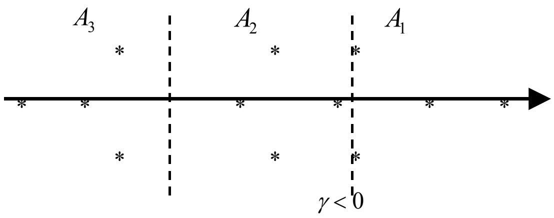

Fig. 3. Discrete spectrum of (2)

Let us now add the following assumptions

- $\sup\{\mathrm{Re}\, s : s \in \lambda\ (A_3) < 0$, $\sup\{\mathrm{Re}\, s : s \in \lambda\ (A_2)\} = \gamma < 0$
- The pair (A_1,B_1) is controllable, The pair (C_1,A_1) is observable
- $\dim X_2 = p \to +\infty \Rightarrow \|B_3\| \to 0$ and $\|C_3\| \to 0$.

The last assumption is fulfilled if, e.g., self-adjoint generator has compact resolvent (the eigenvectors form a basis of the given space).

4. Finite-dimensional stabilizing controller

Let us now consider dynamic feedback Mitkowski [1991, s. 233] of form:

$$\begin{bmatrix} \dot{w}_1(t) \\ \dot{w}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 - G_1 C_1 + B_1 K_1 & -G_1 C_2 \\ B_2 K_1 & A_2 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} y(t), \tag{5}$$

$$u(t) = K_1 w_1(t), \quad w_i(t) \in X_i, \quad i = 1,2.$$

Let us assume that the conditions mentioned in previous section are fulfilled. There exists a finite dimensional stabilizing controller (5), such that the closed-loop system (2) with (5) is exponentially stable with predefined damping coefficient $\alpha \in (\gamma, 0)$. See (Sakawa 1983 [29]) and Mitkowski [1982, 1986, 1988], Mitkowski [1991, s. 230]) for further details.

The design of feedback (5) may be reduced to finding the matrices K_1 and G_1 which can be done using methods known from finite dimensional systems' analysis, e.g., LQ design. The desired damping coefficient $\alpha \in (\gamma, 0)$ can be found by increasing $p = \dim X_2$.

A discrete version of the controller (Mitkowski [1991, s. 236]) can be obtained using formulas (1) and remembering that the system is asymptotically stable if the eigenvalues lie inside the unit circle (assumption 4 and 5, and fig 3). The matrices K_1 i G_1 should be found in a way that guarantees that the eigenvalues of matrices $A_1+B_1K_1$ and $A_1-G_1C_1$ lie inside the unit circle (for example, we can set them as zeros).

5. Stabilization of finite dimensional discrete approximation of the model

For design of closed-loop control system with use of simulation methods, we approximate the system (3) with finite dimensional approximation. We replace the operator A_3 with matrix A_3 of appropriate dimensions (depending on desired accuracy of approximation). This algorithm will be now illustrated with laboratory experiment (Oprzędkiewicz 2003).

The continuous finite dimensional system S(A,B,C,D) with matrices A,B,C, D may de transformed to a discrete system $S^+(A^+, B^+, C^+, D^+)$. In numerical approach, the system S^+ is called differential scheme of continuous system S. The design of stabilizing discrete controller is now equivalent to design a closed loop system with S^+. The parameters of S^+ can de described with Tustin method (Astrom 1990, Bini 2014, see also Mitkowski 1991, p. 142, with Newton-Cotes formula),

$$\begin{aligned} &A^+ = (I + \frac{h}{2}A)(I - \frac{h}{2}A)^{-1}, \quad B^+ = A^{-1}(A^+ - I)B, \\ &C^+ = C, \quad D^+ = D \end{aligned} \tag{6}$$

The Tustin differential scheme has the following property: the discrete system S^+ is asymptotically stable if and only if the system S is asymptotically stable (the eigenvalues of A are in left half-plane). The inverse matrix A^{-1} in (6) exists if the system S is asymptotically stable (but might be weakly damped).

6. Example

Let us now consider the process of heating a thin rod (Oprzędkiewicz [2001, 2003]) depicted in the figure 4.

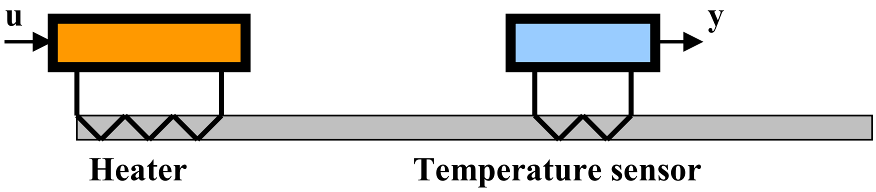

Fig. 4. Heating of a thin rod.

A simplified mathematical model of the analyzed process has the form

$$\begin{aligned} &\frac{\partial x(z,t)}{\partial t} = a\frac{\partial^2 x(z,t)}{\partial z^2} - R_a x(z,t) + b(z)u(t), \quad t \geq 0, \quad z \in [0,1], \\ &\left.\frac{\partial x(z,t)}{\partial z}\right|_{z=0} = \left.\frac{\partial x(z,t)}{\partial z}\right|_{z=1} = 0, \quad t \geq 0, \\ &x(z,0) = 0, \quad z \in (0,1), \\ &y(t) = \int_0^1 c(z)x(z,t)dz. \end{aligned} \tag{7}$$

where

$$b(z)=\begin{cases}1 & for \quad 0\le z\le z_0\\ 0 & for \quad z_0<z\le 1\end{cases}$$

$$c(z)=\begin{cases}\bar{c} & for \quad z_1\le z\le z_2\\ 0 & for \quad 0\le z<z_1 \quad and \quad z_2<z\le 1\end{cases}$$

$$x(z,t)=\sum_{i=0}^{\infty}x_i(t)h_i(z)$$

After the decomposition, we have *S(A,B,C,D)* where

$$A=diag(\lambda_0,\lambda_1,\lambda_2,......),\quad B=[b_0\ b_1\ b_2\ ....]^T,$$
$$C=[c_0\ c_1\ c_2\ ....],\qquad D=0,$$

and

$$X=L^2(0,1;R),\qquad \lambda_i=-i^2\pi^2a-R_a,\quad i=0,1,2,.........$$
$$h_i(z)=\begin{cases}1 & for \quad i=0\\ \sqrt{2}\cos(i\pi z) & for \quad i=1,2,3,.....\end{cases} \tag{8}$$
$$b_i=\int_0^1 b(z)h_i(z)dz,\quad c_i=\int_0^1 c(z)h_i(z)dz,$$

We have the following parameters (6) (verified in a laboratory Oprzędkiewicz 2001):
$a=0.000945$, $R_a=0.0271$, $\bar{c}=25.7922$, $z_0=1/13$, $z_1=25/52$, $z_2=27/52$.
From (7), we have

$A=diag($ -0.0269 -0.0358 -0.0624 -0.1068 -0.1690 -0.2490 -0.3467
-0.4621 -0.5954 -0.7464 -0.9152 -1.1017 -1.3060 -1.5281 -1.7679
-2.0255 -2.3009 -2.5940 -2.9049 -3.2335 -3.5800 -3.9441 -4.3261
-4.7258 -5.1433)

$B=[$0.0769 0.1077 0.1046 0.0995 0.0926 0.0842 0.0745 0.0638
0.0526 0.0412 0.0299 0.0190 0.0090 -0.0000 -0.0077 -0.0139
-0.0187 -0.0218 -0.0234 -0.0235 -0.0223 -0.0200 -0.0168 -0.0130
-0.0087$]^T$

$C=[$1.0171 0 -1.4348 -0.0000 1.4244 -0.0000 -1.4070 0.0000
1.3830 -0.0000 -1.3524 -0.0000 1.3156 -0.0000 -1.2729 -0.0000
1.2246 -0.0000 -1.1711 -0.0000 1.1130 -0.0000 -1.0507 0.0000 0.9848$]$

and $D=0$.
In order to perform the simulation, the heating process was implemented with use of Matlab/Simulink environment (see Fig. 1).

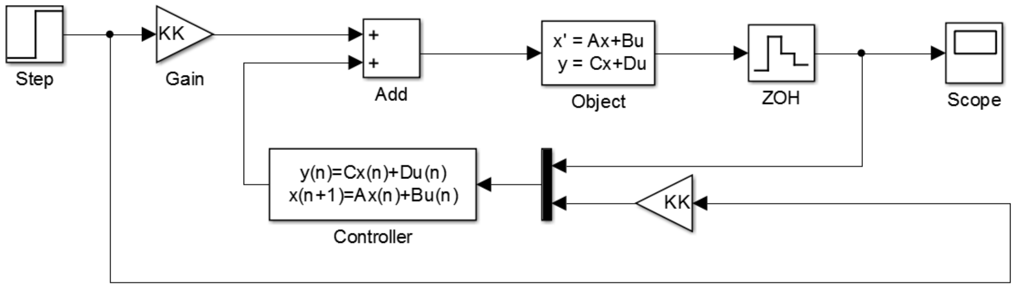

Fig. 5. Simulink system

The zero-order-hold is necessary to simulate a measurement device (e.g. thermometer) with various sampling times. We used the Tustin method (see, e.g., Astrom 1990) to discretize the compensator and then find the appropriate sampling frequency. It transforms the continuous system *S(A,B,C,D)* into a discrete one for a given sampling time *h* using the formulas

$$\begin{aligned} A^{+} &= (I+\frac{h}{2}A)(I-\frac{h}{2}A)^{-1} \\ B^{+} &= A^{-1}(A^{+}-I)B \\ C^{+} &= C \\ D^{+} &= D \end{aligned} \tag{8}$$

During the simulation we wanted to find optimal sampling time of the compensator for various sampling frequencies for temperature measurement. We used the performance indicator proposed by Bini [2014]:

$$J(N)=\frac{1}{N}\int_{0}^{T}|\dot{u}(t)|dt \tag{9}$$

During the simulations, we set T=200 [s]. For optimization, we used golden search with parabolic interpolation implemented in Matlab Optimization Toolbox. The optimization constraints were chosen as $1 \le N \le 10^{6}$. The results are gathered in Table 1.

Table 1

The results of optimization

Temperature sampling frequency [Hz]	Optimal number of samples N_{opt}	Sampling time $h=\frac{T}{N_{opt}}$ [s]
10	23700	0.0084
1	23896	0.0083
0.1	68957	0.0029
0.03	84140	0.0024
0.02	48284	0.0041
0.01	69997	0.0028

It can be seen that sampling time of the controller increases with increasing sampling frequency. This means that we have a buffer in the controller for doing necessary calculations. The accuracy of temperature measurements and controller performance are depicted in the figures 6 and 7.

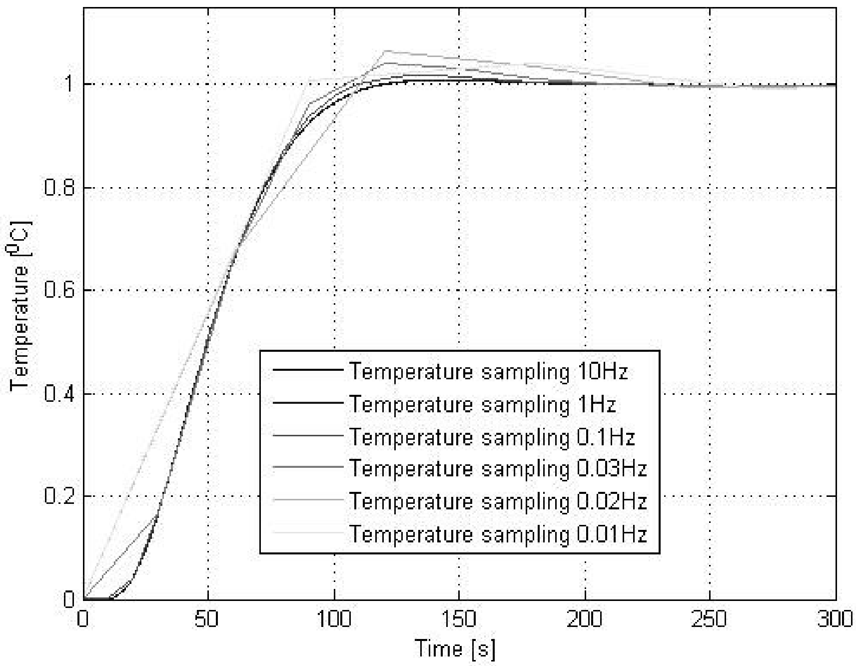

Fig. 6. Temperature for various sampling frequencies

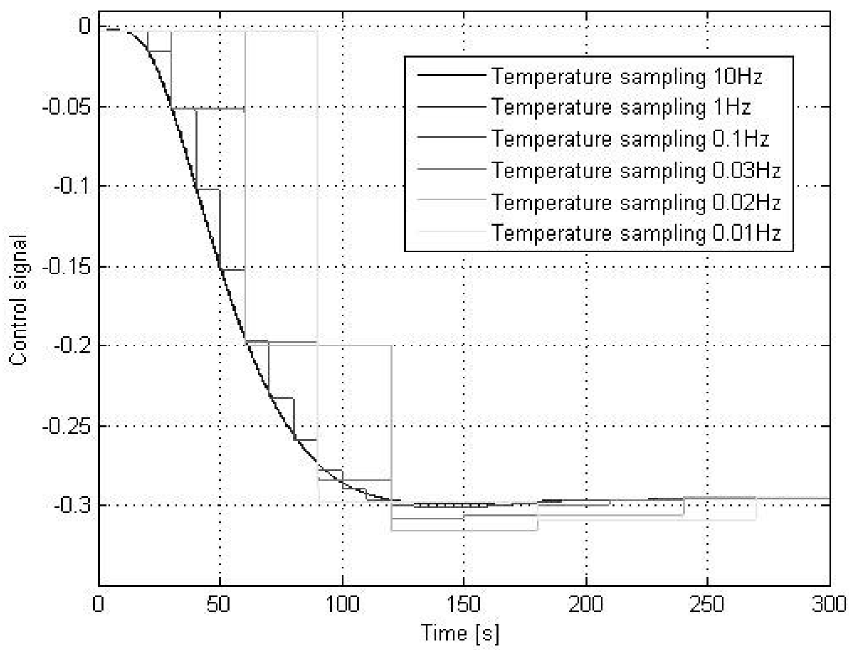

Fig. 7. Control signal for various sampling frequencies

7. Conclusion

In this paper, we presented an algorithm of controller design using two methods: theoretical approach to design finite dimensional feedback (5) and stabilization method based on approximation (Tustin, finite dimensional) – see example (7).

Nevertheless, the proposed algorithm is general and may be used for control of various systems. One of the possible way of applications may be non-integer order diffusion equation Gal and Warma [2016]. However, it will require further analysis and research, as the methods for integer order systems cannot be directly applied to them.

Acknowledgment
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