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STABILNOŚĆ PRZEDZIAŁOWYCH DODATNICH DYSKRETNYCH UKŁADÓW LINIOWYCH RZĘDU CAŁKOWITEGO I NIECAŁKOWITEGO

Streszczenie. W pracy podano analizę stabilności przedziałowych dodatnich układów liniowych dyskretnych opisanych równaniami różnicowymi rzędu całkowitego i niecałkowitego (ułamkowego). Uogólniono twierdzenie Kharitonova (Charitonova) na dodatnie przedziałowe układy liniowe dyskretne rzędu całkowitego i niecałkowitego (ułamkowego).Wykazano, że:

1) przedziałowy dodatni układ $x_{i+1} = Ax_i,\ A \in \Re_+^{n\times n},\ A_1 \le A \le A_2$ jest stabilny asymptotycznie wtedy i tylko wtedy gdy macierze $A_i,\ i = 1,2$ są macierzami Schura,

2) przedziałowy dodatni układ jest stabilny asymptotycznie jeżeli dolne ograniczenia współczynników wielomianu charakterystycznego są dodatnie,

3) przedziałowy dodatni układ dyskretny linowy rzędu niecałkowitego jest stabilny asymptotycznie wtedy i tylko wtedy gdy układ dolnego ograniczenia jest stabilny asymptotycznie.

STABILITY OF INTERVAL POSITIVE STANDARD AND FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS

Summary. The stability of interval positive discrete-time linear systems is addressed. It is shown that:

1) The interval positive system $x_{i+1} = Ax_i,\ A \in \Re_+^{n\times n},\ A_1 \le A \le A_2$ is asymptotically stable if and only if the matrices $A_i,\ i = 1,2$ are Schur matrices.

2) The interval positive system is asymptotically stable if the lower bounds of coefficients of its characteristic polynomial are positive.

3) The interval positive fractional discrete-time linear systems are asymptotically stable if and only if the lower bounds systems are asymptotically stable.

The classical Kharitonov theorem is extended to the interval positive fractional linear systems

1. Introduction

A dynamical system is called positive if its state variables take nonnegative values for all nonnegative inputs and nonnegative initial conditions. The positive linear systems have been investigated in [1, 5, 11] and positive nonlinear systems in [6, 7, 9, 17, 18]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Mathematical fundamentals of the fractional calculus are given in the monographs [24-28]. Fractional dynamical linear and nonlinear systems have been investigated in [6, 8, 10, 13, 15, 18, 28-34].

Positive linear systems with different fractional orders have been addressed in [3, 12, 14, 22,31]. Descriptor (singular) linear systems have been analyzed in [9, 15, 16] and the stability of a class of nonlinear fractional-order systems in [6, 18, 25]. Application of Drazin inverse to analysis of descriptor fractional discrete-time linear systems has been presented in [8].Comparison of three method of analysis of the descriptor fractional systems has been presented in [30]. Stability of linear fractional order systems with delays has been analyzed in [2] and simple conditions for practical stability of positive fractional systems have been proposed in [4]. The stability of interval positive continuous-time and discrete-time linear systems have been addressed in [20, 21].

In this paper the asymptotic stability of interval positive standard and fractional discrete-time linear systems will be addressed.

The paper is organized as follows. In section 2 some basic definitions and theorems concerning positive linear systems and polynomials with interval coefficients are recalled. In section 3 the stability of positive interval linear systems described by the state equation is investigated. In section 4 some basic definitions and theorems concerning positivity and stability of fractional discrete-time linear systems are recalled. Stability of the interval positive fractional linear systems is analyzed in section 5. Convex linear combination of Schur polynomials and the stability of interval positive discrete-time linear systems is investigated in section 6. Concluding remarks are given in section 7.

The following notations will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix, for $A=[a_{ij}]\in\Re^{n\times n}$ and $B=[b_{ij}]\in\Re^{n\times n}$ inequality $A\ge B$ means $a_{ij}\ge b_{ij}$ for $i,j=1,2,...,n$.

2. Preliminaries

Consider the autonomous continuous-time linear system

$$\dot{x}(t)=Ax(t),\ t\ge 0,, \tag{2.1}$$

where $x(t)\in\Re^n$ is the state vector and $A\in\Re^{n\times n}$.

Definition 2.1. [5, 11] The system (2.1) is called positive if $x(t)\in\Re_+^n$, $t\ge 0$ for any initial conditions $x_0=x(0)\in\Re_+^n$.

Theorem 2.1. [5, 11] The system (2.1) is positive if and only if its matrix A is the Metzler matrix.

Definition 2.2. [5, 11] The positive system (2.1) is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \text{ for all finite } x(0) \in \Re^n_+ .$$

Theorem 2.2. [5, 11] The positive system (2.1) is asymptotically stable if and only if one of the equivalent conditions is satisfied:

1) All coefficient of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{2.2}$$

are positive, i.e. $a_k > 0$ for $k = 0,1,\ldots,n-1$.

2) All principal minors $\overline{M}_i$, $i = 1,\ldots,n$ of the matrix $-A$ are positive, i.e.

$$\overline{M}_1 = \left|-a_{11}\right| > 0,\ \overline{M}_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0,\ \ldots,\ \overline{M}_n = \det[-A] > 0 . \tag{2.3}$$

3) There exists strictly positive vector $\lambda^T = [\lambda_1 \quad \Lambda \quad \lambda_n]^T$, $\lambda_k > 0$, $k = 1,\ldots,n$ such that

$$A\lambda < 0 \text{ or } A^T\lambda < 0 . \tag{2.4}$$

If $\det A \neq 0$ then we may choose $\lambda = -A^{-1}c$, where $c \in \Re^n$ is any strictly positive vector.

Consider the set (family) of the *n*-degree polynomials

$$p_n(s) := a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{2.5a}$$

with the interval coefficients

$$\underline{a_i} \le a_i \le \overline{a_i},\ i = 0,1,\ldots,n. \tag{2.5b}$$

Using (2.5) we define the following four polynomials:

$$\begin{aligned} p_{1n}(s) &:= \underline{a_0} + \underline{a_1}s + \overline{a_2}s^2 + \overline{a_3}s^3 + \underline{a_4}s^4 + \underline{a_5}s^5 + \ldots \\ p_{2n}(s) &:= \underline{a_0} + \overline{a_1}s + \overline{a_2}s^2 + \underline{a_3}s^3 + \underline{a_4}s^4 + \overline{a_5}s^5 + \ldots \\ p_{3n}(s) &:= \overline{a_0} + \underline{a_1}s + \underline{a_2}s^2 + \overline{a_3}s^3 + \overline{a_4}s^4 + \underline{a_5}s^5 + \ldots \\ p_{4n}(s) &:= \overline{a_0} + \overline{a_1}s + \underline{a_2}s^2 + \underline{a_3}s^3 + \overline{a_4}s^4 + \overline{a_5}s^5 + \ldots \end{aligned} \tag{2.6}$$

Kharitonov Theorem: The set of polynomials (2.5) is asymptotically stable if and only if the four polynomials (2.6) are asymptotically stable.
Proof is given in [23].

Consider the autonomous discrete-time linear system

$$x_{i+1} = Ax_i,\ i \in Z_+ = \{0,1,\ldots\} \tag{2.7}$$

where $x \in \Re^n$ is the state vector and $A \in \Re^{n\times n}$

Definition 2.3. [5, 11] The discrete-time linear system (2.7) is called (internally) positive if $x_i \in \Re^n_+$, $i \in Z_+$ for any initial conditions $x_0 \in \Re^n_+$.

Theorem 2.3. [5, 11] The discrete-time linear system (2.7) is positive if and only if

$$A \in \Re^{n\times n}_+ \tag{2.8}$$

Definition 2.4. [5, 11] The positive discrete-time linear system (2.7) is called asymptotically stable if

$$\lim_{i\to\infty} x_i = 0 \text{ for any } x_0 \in \Re^n_+ . \tag{2.9}$$

Theorem 2.4. The positive discrete-time linear system (2.7) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficient of the polynomial

$$p_n(z) = \det[I_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 \tag{2.10}$$

are positive, i.e. $a_i > 0$ for $i = 0,1,...,n-1$.

2) All principal minors of the matrix $\overline{A} = I_n - A = [\overline{a_{ij}}]$ are positive, i.e.

$$\overline{M}_1 = |\overline{a_{11}}| > 0,\ \overline{M}_2 = \begin{vmatrix} \overline{a_{11}} & \overline{a_{12}} \\ \overline{a_{21}} & \overline{a_{22}} \end{vmatrix} > 0,\ \ldots,\ \overline{M}_n = \det \overline{A} > 0. \tag{2.11}$$

3) There exists strictly positive vector $\lambda^T = [\lambda_1 \quad \Lambda \quad \lambda_n]$, $\lambda_i > 0$, $i = 1,...,n$ such that

$$A\lambda < \lambda. \tag{2.12}$$

If $[A - I_n] \in M_n$ is asymptotically stable then we may choose $\lambda = [I_n - A]^{-1}c$, where $c \in \Re_+^n$ is strictly positive.

3. Stability of positive interval linear system

Consider the interval positive linear discrete-time system

$$x_{i+1} = Ax_i \tag{3.1}$$

where $x_i \in \Re^n$ is the state vector and the matrix $A \in \Re_+^{n\times n}$ is defined by

$$A_1 \le A \le A_2 \text{ or equivalently } A \in [A_1, A_2] \tag{3.2}$$

Definition 3.1. The interval positive system (3.1) is called asymptotically stable if the system is asymptotically stable for all matrices $A \in \Re_+^{n\times n}$ satisfying the condition (3.2).

By condition (2.12) of Theorem 2.4 the positive system (3.1) is asymptotically stable if and only if there exists strictly positive vector $\lambda > 0$ such that (2.12) holds.

For two positive linear systems

$$x_{1,i+1} = A_1 x_{1,i},\ A_1 \in \Re_+^{n\times n} \tag{3.3a}$$

and

$$x_{2,i+1} = A_2 x_{2,i},\ A_2 \in \Re_+^{n\times n} \tag{3.3b}$$

there exists a strictly positive vector $\lambda \in \Re_+^n$ such that

$$A_1\lambda < \lambda \text{ and } A_2\lambda < \lambda \tag{3.4}$$

if and only if the systems (3.3) are asymptotically stable.

Example 3.1. Consider the positive linear system (3.1) with the matrices

$$A_1 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.3 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.5 \end{bmatrix} \tag{3.5}$$

Note that for $\lambda^T = [1\ 1]$ we have

$$A_1\lambda = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.6 \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3.6a}$$

$$A_2\lambda = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.9 \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3.6b}$$

Therefore, by the condition (2.12) of Theorem 2.4 the positive systems are asymptotically stable.

Theorem 3.1. If the matrices $A_1 \in \Re_+^{n\times n}$ and $A_2 \in \Re_+^{n\times n}$ of positive systems (3.3) are asymptotically stable then their convex linear combination

$$A = (1-k)A_1 + kA_2 \text{ for } 0 \le k \le 1 \tag{3.7}$$

is also asymptotically stable.

Proof. By condition (2.12) of Theorem 2.4 if the positive linear systems (3.3) are asymptotically stable then there exists strictly positive vector $\lambda \in \Re_+^n$ such that (3.4) holds.
Using (3.7) and (3.4) we obtain

$$A\lambda = [(1-k)A_1 + kA_2]\lambda = (1-k)A_1\lambda + kA_2\lambda < (1-k)\lambda + k\lambda = \lambda \text{ for } 0 \le k \le 1 \tag{3.8}$$

Therefore, if the positive linear systems (3.3) are asymptotically stable and (3.4) holds then their convex linear combination is also asymptotically stable. □
Theorem 3.2. The interval positive system (3.1) is asymptotically stable if and only if the positive systems (3.3) are asymptotically stable.
Proof. By condition (2.12) of Theorem 2.4 the matrices $A_1 \in R_+^{n\times n}$, $A_2 \in R_+^{n\times n}$ are asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \Re_+^n$, such that (3.4) holds. The convex linear combination (3.7) satisfies the condition $A\lambda < 0$ if and only if (3.4) holds. Therefore, the interval positive system (3.1) is asymptotically stable if and only if the positive systems (3.3) are asymptotically stable. □
Example 3.2. Consider the interval positive linear system (3.1) with the matrices

$$A_1 = \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.5 \end{bmatrix} \tag{3.9}$$

For the matrices (3.9) we choose $\lambda = [0.95 \;\; 0.85]^T$ and we obtain

$$\begin{aligned} A_1\lambda &= \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}\begin{bmatrix} 0.95 \\ 0.85 \end{bmatrix} = \begin{bmatrix} 0.56 \\ 0.445 \end{bmatrix} < \begin{bmatrix} 0.95 \\ 0.85 \end{bmatrix} \\ A_2\lambda &= \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.5 \end{bmatrix}\begin{bmatrix} 0.95 \\ 0.85 \end{bmatrix} = \begin{bmatrix} 0.93 \\ 0.805 \end{bmatrix} < \begin{bmatrix} 0.95 \\ 0.85 \end{bmatrix} \end{aligned} \tag{3.10}$$

Therefore, by Theorem 3.2 the interval positive system (3.1) with (3.9) is asymptotically stable.

4. Fractional discrete-time systems

Consider the autonomous fractional discrete-time linear system

$$\Delta^\alpha x_{i+1} = Ax_i, \; 0 < \alpha < 1, \; i \in Z_+, \tag{4.1}$$

where

$$\Delta^\alpha x_i = \sum_{j=1}^{i} c_j x_{i-j}, \tag{4.2a}$$

$$c_j = (-1)^j \binom{\alpha}{j}, \; \binom{\alpha}{j} = \begin{cases} 1 & \text{for} \quad j = 0 \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for} \quad j = 1,2,\ldots \end{cases} \tag{4.2b}$$

is the fractional α-order difference of x_i and $x_i \in \Re^n$, $u_i \in \Re^m$ are the state and input vectors and $A \in \Re^{n\times n}$.
Substitution of (4.2) into (4.1) yields

$$x_{i+1} = A_\alpha x_i - \sum_{j=2}^{i+1} c_j x_{i-j+1}, \; i \in Z_+, \tag{4.3a}$$

where

$$A_\alpha = A + I_n\alpha. \tag{4.3b}$$

Lemma 4.1.[19] If $0<\alpha<1$ then

1) $-c_j>0$ for $j=1,2,...$ (4.4a)

2) $\sum_{j=1}^{n} c_j=-1$. (4.4b)

Definition 4.1. [19] The fractional system (4.1) is called (internally) positive if $x_i\in\Re_+^n$, $i\in Z_+$ for any initial conditions $x_0\in\Re_+^n$.

Theorem 4.1. [19] The fractional system (4.1) is positive if and only if

$$A_\alpha\in M_n. \tag{4.5}$$

Definition 4.4. The fractional positive system (4.1) is called asymptotically stable if

$$\lim_{i\to\infty} x_i=0 \text{ for all } x_0\in\Re_+^n. \tag{4.6}$$

Theorem 4.4. [19] The fractional positive system (4.1) is asymptotically stable if and only if one of the equivalent conditions is satisfied:

1) All coefficient of the characteristic polynomial

$$p_A(z)=\det[I_n(z+1)-A]=z^n+a_{n-1}z^{n-1}+...+a_1z+a_0 \tag{4.7}$$

are positive, i.e. $a_k>0$ for $k=0,1,...,n-1$.

2) All principal minors of the matrix

$$\bar{A}=I_n-A=\begin{bmatrix}\bar{a}_{11} & ... & \bar{a}_{1n}\\ \mathrm{M} & ... & \mathrm{M}\\ \bar{a}_{n1} & ... & \bar{a}_{nn}\end{bmatrix} \tag{4.8}$$

are positive, i.e.

$$|a_{11}|>0,\ \begin{vmatrix}\bar{a}_{11} & \bar{a}_{12}\\ \bar{a}_{21} & \bar{a}_{22}\end{vmatrix}>0,\ \dots,\ \det\bar{A}>0. \tag{4.9}$$

3) There exists strictly positive vector $\lambda^T=[\lambda_1 \quad \Lambda \quad \lambda_n]^T$, $\lambda_k>0$, $k=1,...,n$ such that

$$[A-I_n]\lambda<0. \tag{4.10}$$

Theorem 4.3. The fractional positive system (4.1) with (4.3b) is asymptotically stable if and only if there exists a strictly positive vector $\lambda>0$ such that

$$.\quad A\lambda<0 \tag{4.11}$$

Proof. Note that the positive fractional system (4.3) can be considered as a positive linear system with increasing to infinity numbers of delays. It is well-known [19] that the stability of positive discrete-time linear systems depends only on the sum of state matrices

$$\hat{A}=A_\alpha-\sum_{j=2}^{\infty} c_j I_n, \tag{4.12}$$

From (4.4b) we have

$$-\sum_{j=2}^{\infty} c_j=1-\alpha. \tag{4.13}$$

Substituting (4.13) into (4.12) we obtain

$$\hat{A}=A_\alpha+(1-\alpha)I_n=A+I_n, \tag{4.14}$$

since $A_\alpha=A+I_n\alpha$.

Applying the condition (4.10) to (4.14) we obtain (4.11). □

Example 4.1. Consider the fractional discrete-time system (4.1) for $\alpha = 0.6$ with the matrix

$$A = \begin{bmatrix} -0.4 & 0.2 \\ 0.3 & -0.5 \end{bmatrix}. \tag{4.15}$$

The fractional system is positive since the matrix

$$A_\alpha = A + I_2\alpha = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.1 \end{bmatrix} \in \Re_+^{2\times 2} \tag{4.16}$$

has positive entries.

The positive fractional system is asymptotically stable since for $\lambda^T = [1 \quad 1]$ we have

$$A\lambda = \begin{bmatrix} -0.4 & 0.2 \\ 0.3 & -0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} < 0 \tag{4.17}$$

and the condition (4.11) is satisfied.

5. Fractional interval positive linear continuous-time systems

Consider the interval fractional positive discrete-time linear system (4.1) with the interval matrix $A \in \Re_+^{n\times n}$ defined by

$$A_1 \le A \le A_2 \text{ or equivalently } A \in [A_1, A_2]. \tag{5.1}$$

Definition 5.1. The interval fractional positive system with (5.1) is called asymptotically stable if the system is asymptotically stable for all matrices $A \in \Re_+^{n\times n}$ belonging to the interval $[A_1, A_2]$.

By condition (4.11) of Theorem 4.3 the interval fractional positive system is asymptotically stable if and only if there exists strictly positive vector $\lambda > 0$ such that $A\lambda < 0$ for all $A \in [A_1, A_2]$.

Definition 5.2. The matrix

$$A = (1-k)A_1 + kA_2, \; 0 \le k \le 1, \; A_1 \in \Re^{n\times n}, \; A_2 \in \Re^{n\times n} \tag{5.2}$$

is called the convex linear combination of the matrices A_1 and A_2.

Theorem 5.1. The convex linear combination (5.2) is asymptotically stable if and only if the matrices $A_1 \in \Re^{n\times n}$ and $A_2 \in \Re^{n\times n}$ are asymptotically stable.

Proof. If the matrices $A_1 \in \Re^{n\times n}$ and $A_2 \in \Re^{n\times n}$ are asymptotically stable then by condition (4.11) of Theorem 4.3 there exists strictly positive vector $\lambda \in \Re_+^n$ such that

$$A_l\lambda < 0 \text{ for } l = 1,2. \tag{5.3}$$

In this case using (5.2) and (5.3) we obtain

$$A\lambda = [(1-k)A_1 + kA_2]\lambda = (1-k)A_1\lambda + kA_2\lambda < 0 \text{ for } 0 \le k \le 1. \tag{5.4}$$

Therefore, if the matrices A_l, l = 1,2 are asymptotically stable then the convex linear combination (5.2) is also asymptotically stable. Necessity follows immediately from the fact that k can be equal to zero and one. □

Theorem 5.2. The interval fractional positive system (4.1) with (5.1) is asymptotically stable if and only if the matrices $A_1 \in \Re^{n\times n}$ and $A_2 \in \Re^{n\times n}$ are Schur matrices.

Proof. By condition (4.11) of Theorem 4.3 the matrices $A_1 \in \Re^{n\times n}$ and $A_2 \in \Re^{n\times n}$ are Schur matrices if and only if there exists strictly positive vector $\lambda \in \Re_+^n$ such that (5.3) holds. The convex linear combination (5.2) satisfies the condition $A\lambda < 0$ if and only if

(5.3) holds. Therefore, the interval fractional positive systems (4.1) with (5.1) is asymptotically stable if and only if $A_1 \in \Re^{n\times n}$ and $A_2 \in \Re^{n\times n}$ are Schur matrices. □

Example 5.1. Consider the interval fractional positive linear systems (4.1) with the matrices

$$A_1 = \begin{bmatrix} -0.3 & 0.1 \\ 0.05 & -0.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.5 & 0.3 \\ 0.2 & -0.6 \end{bmatrix} \tag{5.5}$$

It is easy to check that for $\lambda^T = [1 \;\; 1]$ we have

$$\begin{aligned} A_1\lambda &= \begin{bmatrix} -0.3 & 0.1 \\ 0.05 & -0.4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.2 \\ -0.35 \end{bmatrix} < 0 \\ A_2\lambda &= \begin{bmatrix} -0.5 & 0.3 \\ 0.2 & -0.6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.2 \\ -0.4 \end{bmatrix} < 0 \end{aligned} \tag{5.6}$$

Therefore, by Theorem 5.2 the interval fractional positive system (4.1) with (5.1) is asymptotically stable.

6. Convex linear combination of Schur polynomials and stability of interval fractional positive linear systems

Definition 6.1. The polynomial

$$p(z) = b_n z^n + b_{n-1}z^{n-1} + \ldots + b_1 z + b_0 \tag{6.1}$$

is called Schur polynomial if its zeros z_l, $l = 1,\ldots,n$ satisfy the condition

$$|z_l| < 1 \text{ for } l = 1,\ldots,n. \tag{6.2}$$

Definition 6.2. The polynomial

$$p(z) = (1-k)p_1(z) + kp_2(z) \text{ for } k \in [0,1] \tag{6.3}$$

is called convex linear combination of the polynomials

$$p_i(z) = b_{i,n}z^n + b_{i,n-1}z^{n-1} + \ldots + b_{i,1}z + b_{i,0}, \; i = 1,2. \tag{6.4}$$

Theorem 6.1.[21] The convex linear combination of the Hurwitz polynomials is also a Hurwitz polynomial.

For positive linear systems we have the following relationship between Hurwitz and Schur polynomials.

Theorem 6.2. The polynomial

$$p(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{6.5}$$

is Hurwitz and the polynomial

$$p(z) = b_n z^n + b_{n-1}z^{n-1} + \ldots + b_1 z + b_0 \tag{6.6}$$

is Schur polynomial if and only if their coefficients a_i and b_i $i = 0,1,\ldots,n$ are related by

$$\begin{aligned} a_0 &= b_0 + b_1 + \ldots + b_n, \\ a_1 &= b_1 + 2b_2 + \ldots + nb_n, \\ &\vdots \\ a_{n-1} &= b_{n-1} + nb_n, \\ a_n &= b_n. \end{aligned} \tag{6.7}$$

Proof. It is well-known [19] that for positive linear discrete-time and continuous-time systems the zeros z_l, $l=1,...,n$ of the polynomial (6.6) and the zeros s_l, $l=1,...,n$ of the polynomial (6.5) are related by

$$z_l = s_l + 1,\ l=1,...,n. \tag{6.8}$$

Substituting $z=s+1$ into the polynomial (6.6) we obtain

$$b_n(s+1)^n + b_{n-1}(s+1)^{n-1} + ... + b_1(s+1) + b_0 = a_n s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \tag{6.9}$$

and it is easy to verify that the coefficients a_i and b_i $i=0,1,...,n$ are related by (6.7).
The polynomial (6.5) is Hurwitz if and only if $a_i > 0$ for $i=0,1,...,n$ and the polynomial (6.6) is Schur if and only if $b_i > 0$ for $i=0,1,...,n$. From (6.7) it follows that $b_i > 0$, $i=0,1,...,n$ implies $a_i > 0$ for $i=0,1,...,n$. □

Example 6.1. The polynomial

$$p(z) = z^2 + 0.6z + 0.08 \tag{6.10}$$

of positive discrete-time linear system is Schur polynomial since its zeros are: $z_1=-0.2,\ z_2=-0.4$.
Substituting $z=s+1$ into (6.10) we obtain

$$p(s) = (s+1)^2 + 0.6(s+1) + 0.08 = s^2 + 2.6s + 1.68 \tag{6.11}$$

with the zeros $s_1=-1.2,\ s_2=-1.4$. Therefore, the polynomial (6.11) is Hurwitz.

Theorem 6.3. The interval positive fractional discrete-time linear system with the characteristic polynomial (6.6) is asymptotically stable if the lower $\underline{b}_i$ and the upper $\overline{b}_i$, $i=0,1,...,n$ bounds of its coefficients are positive.

Proof. From (6.7) it follows that $\underline{b}_i > 0$, $i=0,1,...,n$ implies $a_i > 0$ for $i=0,1,...,n$ and the characteristic polynomial (6.5) is Hurwitz. By Theorem 2.2 the continuous-time system is asymptotically stable. Similar result we obtain for the upper bound. Therefore, the interval fractional positive discrete-time system (6.6) is asymptotically stable if the lower and upper bound of the coefficients are positive. □

Remark. 6.1. The equalities (6.7) can be used to compute the lower and upper bounds of the coefficients a_i, $i=0,1,...,n$ of polynomial (6.5) knowing the lower and upper bounds of the coefficients b_i, $i=0,1,...,n$ of polynomial (6.6).

Example 6.2. Consider the characteristic polynomial

$$p(z) = b_2 z^2 + b_1 z + b_0 \tag{6.12}$$

of positive fractional discrete-time systems with the interval coefficients

$$1 \le b_2 \le 3,\ 2 \le b_1 \le 3,\ 1 \le b_0 \le 4. \tag{6.13}$$

The equivalent characteristic polynomial of continuous-time system has the form

$$p(s) = b_2(s+1)^2 + b_1(s+1) + b_0 = a_2 s^2 + a_1 s + a_0 \tag{6.14}$$

where

$$a_2 = b_2,\ a_1 = b_1 + 2b_2,\ a_0 = b_0 + b_1 + b_2. \tag{6.15}$$

Therefore, the interval coefficients of characteristic polynomial of continuous-time system are

$$1 \le a_2 \le 3,\ 4 \le a_1 \le 9,\ 4 \le a_0 \le 10. \tag{6.16}$$

By Theorem 6.3 the interval positive discrete-time linear system with (6.12) is asymptotically stable since the lower bounds (6.16) are positive.

7. Concluding remarks

The asymptotic stability of interval positive linear discrete-time systems has been addressed. It has been shown that:
The interval positive system (3.1) is asymptotically stable if and only if the matrices A_i, $i=1,2$ are Schur matrices (Theorem 3.2, 5.2). The convex linear combination of the Hurwitz polynomials is also the Hurwitz polynomial (Theorems 6.1).

The interval positive system is asymptotically stable if the lower bounds of coefficients of the polynomial of system are positive (Theorem 6.3). The considerations have been illustrated by numerical examples of positive interval discrete-time systems. The considerations can be extended to continuous-time positive standard and fractional linear systems. An open problem is an extension of the considerations to nonpositive standard and fractional discrete-time and continuous-time linear systems.

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